Aspects of Diffeomorphism Invariant Theory of Extended Objects

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The structure of a diffeomorphism invariant Lagrangians for an extended object W embedded in a bulk space M is discussed by following a close analogy with the relativistic particle in electromagnetic field as a system that is reparametrization-invariant. The current construction naturally contains, relativistic point particle, string theory, and Dirac–Nambu–Goto Lagrangians with Wess–Zumino terms. For Lorentzian metric field, the non-relativistic theory of an integrally submerged W-brane is well defined provided that the brane does not alter the background interaction fields. A natural time gauge is fixed by the integral submergence (sub-manifold structure) within a Lorentzian signature structure. A generally covariant relativistic theory for the discussed brane Lagrangians is also discussed. The mass-shell constraint and the Klein–Gordon equation are shown to be universal when gravity-like interaction is present. A construction of the Dirac equation for the W-brane that circumvents some of the problems associated with diffeomorphism invariance of such Lagrangians by promoting the velocity coordinates into a non-commuting gamma variables is presented.

Keywords: diffeomorphism invariant systems, reparametrization-invariant systems, matter Lagrangian, homogeneous singular Lagrangians, relativistic particle, Dirac equation, string theory, extended objects, branes, interaction fields, generally covariant theory, gauge symmetries, background free theories.

Introduction. The Hamiltonian and Lagrangian formulation[1, 2] are two very useful approaches in physics. In general, these two approaches are related by the Legander transformation. For a reparametrization-invariant theory, however, there are problems in changing from the Lagrangian to the Hamiltonian approach.[2, 3, 4, 5] In this paper the focus is on the properties of reparametrization-invariant matter systems such as, the relativistic particle and its extended object (brane) generalization within the Lagrangian approach. We try to answer the question: "What is the Lagrangian for an extended 'matter' object?"

Matter Lagrangian for relativistic particle. The action for a massive relativistic particle has a nice geometrical meaning: it is the distance along the particle trajectory[6] provided that the units are such that $x^0 = ct$ and the particle moves with a constant 4-velocity ($g_{\mu\nu}v^\mu v^\nu = 1$):

$$S_1 = \int d\tau L_1(x, v) = \int d\tau \sqrt{g_{\mu\nu}v^\mu v^\nu} \rightarrow \int d\tau. \tag{1}$$

For a massless particle, such as a photon, the length of the 4-velocity is zero ($g_{\mu\nu}v^\mu v^\nu = 0$) and the appropriate 'good' action[6] is:

$$S_2 = \int L_2(x, v)d\tau = \int g_{\mu\nu}v^\mu v^\nu d\tau. \tag{2}$$

The Euler–Lagrange equations obtained from S_1 and S_2 are equivalent, even more, they are equivalent to the geodesic equation as well:

$$\frac{d}{d\tau}\vec{v} = D_{\vec{v}}\vec{v} = v^\beta \nabla_\beta \vec{v} = 0 \tag{3}$$

Since the Levi–Civita connection ∇ preserves the length of the vectors[6] ($\nabla g(\vec{v}, \vec{v}) = 0$) this equivalence is not surprising because the Lagrangians in (1) and (2) are functions of the preserved arc length $g(\vec{v}, \vec{v}) = \vec{v}^2$. However, the equivalence between S_1 and S_2 has a much deeper roots.

Homogeneous Lagrangians. Since L_2 is a homogeneous function of order 2 with respect to $\vec{v}$, the corresponding Hamiltonian function ($h = v\partial L/\partial v - L$) is exactly equal to L_2. Thus L_2 is conserved, and so is the length of $\vec{v}$. Any

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homogeneous Lagrangian in $\vec{v}$ of order $n \neq 1$ is conserved because $h = (n-1)L$. If $dL/d\tau = 0$, then the Euler–Lagrange equations for L and $\tilde{L} = f(L)$ are equivalent. This is an equivalence that applies to homogeneous Lagrangians in particular. It is different from the usual equivalence $L \to \tilde{L} = L + d\Lambda/d\tau$ or the more general equivalence discussed by Hojman and Harleston Ref. [7]. Any solution of the Euler–Lagrange equation for $\tilde{L} = L^{\alpha}$ would conserve $L = L_1$ since $\tilde{h} = (\alpha - 1)L^{\alpha}$. All these solutions are solutions of the Euler–Lagrange equation for L as well, thus $L^{\alpha} \subset L$. In general, conservation of L_1 is not guaranteed since $L_1 \to L_1 + d\Lambda/d\tau$ is also a homogeneous Lagrangian of order one equivalent to L_1. This suggests that there may be a choice of Λ, a "gauge fixing", such that $L_1 + d\Lambda/d\tau$ is conserved even if L_1 is not. The above discussion applies to a more general homogeneous Lagrangians as well.[8]

In the example of the relativistic particle, the Lagrangian and the trajectory parameterization have a geometrical meaning. In general, however, parameterization of a trajectory is quite arbitrary for any observer. If there is no preferred trajectory parameterization in a smooth space-time, then we are free to choose the standard of distance (time, using natural units $c = 1$). Thus, *our theory should not depend on the choice of parameterization.* By inspection of the Euler–Lagrange equations, any homogeneous Lagrangian of order n ($L(x, \alpha\vec{v}) = \alpha^n L(x, \vec{v})$) provides a reparametrization invariant equations ($\tau \to \tau/\alpha, \vec{v} \to \alpha\vec{v}$). Next, note that the action S involves an integration that is a natural structure for orientable manifolds (M) with an n-form of the volume. Since a trajectory is a one-dimensional object, then what we are looking at is an embedding $\phi : \mathbb{R}^1 \to M$. This means that we push forward the tangential space $\phi_* : T(\mathbb{R}^1) = \mathbb{R}^1 \to T(M)$, and pull back the cotangent space $\phi^* : T(\mathbb{R}^1) = \mathbb{R}^1 \leftarrow T^*(M)$. Thus a 1-form ω on M that is in $T^*(M)$ ($\omega = A_\mu(x)\, dx^\mu$) will be pulled back on $\mathbb{R}^1$ ($\phi^*(\omega)$) and there it should be proportional to the volume form on $\mathbb{R}^1$ ($\phi^*(\omega) = A_\mu(x)\,(dx^\mu/d\tau)d\tau \sim d\tau$), allowing us to integrate $\int \phi^*(\omega)$:

$$\int \phi^*(\omega) = \int L d\tau = \int A_\mu(x)\, v^\mu d\tau.$$

Therefore, by selecting a 1-form $\omega = A_\mu(x)\, dx^\mu$ on M and using $L = A_\mu(x)\, v^\mu$ we are actually solving for the embedding $\phi : \mathbb{R}^1 \to M$ using a chart on M with coordinates $x : M \to \mathbb{R}^n$. The Lagrangian obtained this way is homogeneous of first order in v with a very simple dynamics. The corresponding Euler–Lagrange equation is $F_{\nu\mu}v^\mu = 0$ where F is a 2-form ($F = dA$) – the Faraday's tensor. If the assumption that L is a pulled back 1-form is relaxed and instead one assumes that it is just a homogeneous Lagrangian of order one, then the corresponding reparametrization-invariant theory may have an interesting dynamics.

First order homogeneous Lagrangians – canonical form. Now we define what we mean by the *canonical form of the first order homogeneous Lagrangian* and why do we prefer this mathematical expression. Let $S_{\alpha_1\alpha_2\ldots\alpha_n}$ be a symmetric tensor of rank n which defines a homogeneous function of order n ($S_n(\vec{v}, ..., \vec{v}) = S_{\alpha_1\alpha_2\ldots\alpha_n} v^{\alpha_1}...v^{\alpha_n}$). The symmetric tensor of rank two is denoted by $g_{\alpha\beta}$. Using this notation, the canonical form of the first order homogeneous Lagrangian is defined as:

$$L(\vec{x}, \vec{v}) = \sum_{n=1}^{\infty} \sqrt[n]{S_n(\vec{v}, ..., \vec{v})} = A_\alpha v^\alpha + \sqrt{g_{\alpha\beta} v^\alpha v^\beta} + ... \sqrt[m]{S_m(\vec{v}, ..., \vec{v})}. \tag{4}$$

Any Lagrangian for the matter should involve interaction fields that couple with the velocity $\vec{v}$ to a scalar. When the matter action is combined with the action for the interaction fields ($\mathcal{S} = \int \mathcal{L} dV$), one obtains a full *background independent theory.* Then the corresponding Euler–Lagrange equations contain "dynamical derivatives" on the left hand side and sources on the right hand side:

$$\partial_\gamma \left(\frac{\delta \mathcal{L}}{\delta(\partial_\gamma \Psi^\alpha)} \right) = \frac{\delta \mathcal{L}}{\delta \Psi^\alpha} + \frac{\partial L_{matter}}{\partial \Psi^\alpha}.$$

The advantage of the canonical form of the first order homogeneous Lagrangian (4) is that each interaction field, which is associated with a symmetric tensor, has a unique matter source that is a monomial in the velocities:

$$\frac{\partial L}{\partial S_{\alpha_1\alpha_2\ldots\alpha_n}} = \frac{1}{n} \left(S_n(\vec{v}, ..., \vec{v}) \right)^{\frac{1-n}{n}} v^{\alpha_1}...v^{\alpha_n}. \tag{5}$$

There are many other ways one can write first-order homogeneous functions.[4] For example, one can consider the following expression $L(\vec{x}, \vec{v}) = \left(h_{\alpha\beta} v^\alpha v^\beta\right) \left(g_{\alpha\beta} v^\alpha v^\beta\right)^{-1/2}$. However, each of the fields h and g has the same source type ($\sim v^\alpha v^\beta$). At this stage, however, it is not clear why the same source type should produce different fields. Therefore, the canonical form (4) seems more appropriate for the current discussion.

Extended objects. In the previous sections, the classical mechanics of a point-like particle have been discussed as a problem concerned with the embedding $\phi : \mathbb{R}^1 \to M$. The map ϕ provides the trajectory (the word line) of the

particle in the target space M. In this sense, we are dealing with a one dimensional object, the world-line of the particle (one dimensional W-brane). We think of an extended object as a manifold W with dimension denoted by D. In this sense, we have to solve for $\phi : W \rightarrow M$ such that some action integral is minimized. From this point of view, we are dealing with mechanics of a brane. In other words, how is this D-dimensional extended object submerged in M, and what are the relevant interaction fields? Following the relativistic point particle discussion, we consider the space of the D-forms over the manifold M, denoted by $\Lambda^D(M)$, that has dimension $\binom{m}{D} = \frac{m!}{D!(m-D)!}$. An element Ω in $\Lambda^D(M)$ has the form $\Omega = \Omega_{\alpha_1 \ldots \alpha_m} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \ldots dx^{\alpha_m}$. We use the label Γ to index different D-forms over $M, \Gamma = 1, 2, \ldots, \binom{m}{D}$; thus $\Omega \rightarrow \Omega^\Gamma = \Omega^\Gamma_{\alpha_1 \ldots \alpha_m} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \ldots dx^{\alpha_m}$. Next we introduce "*generalized velocity vectors*" with components ω^Γ :

$$\omega^\Gamma = \frac{\Omega^\Gamma}{dz} = \Omega^\Gamma_{\alpha_1 \ldots \alpha_D} \frac{\partial \left(x^{\alpha_1} x^{\alpha_2} \ldots x^{\alpha_D}\right)}{\partial (z^1 z^2 \ldots z^D)}, \quad dz = dz^1 \wedge dz^2 \wedge \ldots \wedge dz^D.$$

In the above expression, $\frac{\partial(x^{\alpha_1} x^{\alpha_2} \ldots x^{\alpha_D})}{\partial(z^1 z^2 \ldots z^D)}$ represents the Jacobian of the transformation from coordinates $\{x^\alpha\}$ over the manifold M to coordinates $\{z^a\}$ over the brane. The pull back of a D-form Ω^Γ must be proportional to the volume form over the brane:

$$\phi^* \left(\Omega^\Gamma\right) = \omega^\Gamma dz^1 \wedge \ldots \wedge dz^D = \Omega^\Gamma_{\alpha_1 \ldots \alpha_D} \frac{\partial \left(x^{\alpha_1} \ldots x^{\alpha_D}\right)}{\partial (z^1 \ldots z^D)} dz^1 \wedge \ldots \wedge dz^D.$$

Thus, it is suitable for integration over the W-manifold and the action is:

$$S[\phi] = \int_W L\left(\vec{\phi}, \vec{\omega}\right) dz = \int_W \phi^*(\Omega) = \int_W A_\Gamma(\vec{\phi}) \omega^\Gamma dz.$$

This is a homogeneous function in ω and is reparametrization (diffeomorphism) invariant with respect to the diffeomorphisms of the W-manifold. If we relax the linearity $L(\vec{\phi}, \vec{\omega}) = \phi^*(\Omega) = A_\Gamma(\vec{\phi})\omega^\Gamma$ in $\vec{\omega}$, then the canonical expression for the first order homogeneous Lagrangian is:

$$L\left(\vec{\phi}, \vec{\omega}\right) = \sum_{n=1}^{\infty} \sqrt[n]{S_n(\vec{\omega}, \ldots, \vec{\omega})} = A_\Gamma \omega^\Gamma + \sqrt{g_{\Gamma_1 \Gamma_2} \omega^{\Gamma_1} \omega^{\Gamma_2}} + \ldots \tag{6}$$

At this point, there is a strong analogy between the relativistic point particle and the extended object. Some specific examples of W-brane theories correspond to the following familiar Lagrangians:

Lagrangian for a relativistic point particle in an electromagnetic field: $\dim W = 1$ (World-line) with $\omega^\Gamma \rightarrow v^\alpha = \frac{dx^\alpha}{d\tau}$

$$L\left(\vec{\phi}, \vec{\omega}\right) = A_\Gamma \omega^\Gamma + \sqrt{g_{\Gamma_1 \Gamma_2} \omega^{\Gamma_1} \omega^{\Gamma_2}} \rightarrow q A_\alpha v^\alpha + m \sqrt{g_{\alpha\beta} v^\alpha v^\beta}.$$

Lagrangian for strings: $\dim W = 2$ (World-sheet)
$L\left(x^\alpha, \partial_i x^\beta\right) = \sqrt{Y^{\alpha\beta} Y_{\alpha\beta}}$, with the following notation:

$$\omega^\Gamma \rightarrow Y^{\alpha\beta} = \frac{\partial(x^\alpha, x^\beta)}{\partial(\tau, \sigma)} = \det \begin{pmatrix} \partial_\tau x^\alpha & \partial_\sigma x^\alpha \\ \partial_\tau x^\beta & \partial_\sigma x^\beta \end{pmatrix} = \partial_\tau x^\alpha \partial_\sigma x^\beta - \partial_\sigma x^\alpha \partial_\tau x^\beta.$$

Dirac–Nambu–Goto Lagrangian (DNG)[9]: $L\left(x^\alpha, \partial_W x^\beta\right) = \sqrt{Y^\Gamma Y_\Gamma}$.

The corresponding electromagnetic interaction term for W-banes is know as Wess–Zumino term[10] in string theory.

From the expressions (4) and (6), one can see that the corresponding matter Lagrangians (L), in their canonical form, contain electromagnetic (A) and gravitational (g) interactions, as well as interactions that are not clearly identified yet ($S_n, n > 2$), if present at all in nature. At this stage, we have a theory with background fields since we don't know the equations for the interaction fields A, g, and S_n. To complete the theory, we need to introduce actions for these interaction fields. If one is going to study the new interaction fields $S_n, n > 2$, then some guiding principles for writing field Lagrangians are needed.

One approach is to apply the above discussion and view the S_n fields as related to an embedding of the M-manifold into the manifold of symmetric tensors over M. Another approach would be to use the external derivative d, external multiplication $\wedge$, and Hodge dual $*$ operations in the external algebra $\Lambda(T^*M)$ over M to construct objects

proportional to the volume form over M. For example, for any n-form (A) the expressions $A\wedge *A$ and $dA\wedge *dA$ are forms proportional to the volume form. The next important principle comes from the symmetry in the matter equation. That is, if there is a transformation $A\to A'$ that leaves the matter equations unchanged, then there is no way to distinguish A and A'. Thus the action for the field A should obey the same gauge symmetry. For the electromagnetic field $(A\to A'=A+\mathrm{d}f)$ this leads uniquely to the field Lagrangian $\mathcal{L}=dA\wedge *dA=F\wedge *F$, when for gravity[11] it leads to the Cartan–Einstein action[12] $S\left[R\right]=\int R_{\alpha\beta}\wedge *(\mathrm{d}x^{\alpha}\wedge dx^{\beta})$.

Non-relativistic limit. For a W-brane we assume the existence a local coordinate frame where one component of the generalized velocity can be set to 1 $(\omega^0=1)$. This generalized velocity component is associated with the brane "time coordinate." In fact, $\omega^0=1$ means that there is an integral embedding of the brane in the target space M, and the image of the brane is a sub-manifold of M. If the coordinates of M are labeled so that $x^i=z^i, i=1,...,D$, then x^i are internal coordinates that can collapse to only one coordinate – the "world line". This provides a gauge-fixing that allows one to do canonical quantization. This approach is mainly concerned with the choice of a coordinate time that is used as the trajectory parameter.[13, 14, 15, 16] Such choice removes the reparametrization invariance of the theory.

In a local coordinate system where $\omega^0=1$ and the metric is a "one-time-metric" we have:

$$
\begin{aligned}
L &= A_{\Gamma}\omega^{\Gamma}+\sqrt{g_{\Gamma_1\Gamma_2}\omega^{\Gamma_1}\omega^{\Gamma_2}}+...+\sqrt[m]{S_m\left(\vec{\omega},...,\vec{\omega}\right)}\to\\
&\to A_0+A_i\omega^i+\sqrt{1-g_{ii}\omega^i\omega^i}+...\approx A_0+A_i\omega^i+1-\frac{1}{2}g_{ii}\omega^i\omega^i+....
\end{aligned}
$$

Thus the Hamiltonian function is not zero anymore, so we can do canonical quantization, and the Hilbert space consists of the functions $\Psi\left(x\right)\to\Psi\left(z,\tilde{x}\right)$ where $\tilde{x}=x^i, i=D+1,...,m$. The brane coordinates z should be treated as t in quantum mechanics in the sense that the scalar product should be an integral over the space coordinates $\tilde{x}$. For W-branes the one-time coordinate reflects separation of the internal from the external coordinates when the W-brane is considered as a sub-manifold of the target space manifold M.

Even though canonical quantization can be applied after the above gauge fixing, one is not usually happy because the covariance of the theory is lost and time is a privileged coordinate. In general, there are well developed procedures for covariant quantization.[13, 15, 17, 18, 19] In this paper, however, we are not going to discuss these methods. Instead, we will employ a different quantization strategy[20], but before that we will discuss the mass-shell and Klein–Gordon equations.

The mass-shell and Klein–Gordon equation. Since the functional form of the canonical Lagrangian is the same for any W-brane, we use v, but it could be ω as well. We define the momentum p and generalized momentum π for our canonical Lagrangian as follow:

$$
\begin{aligned}
p_{\Gamma} &= \frac{\delta L\left(\phi,\omega\right)}{\delta\omega^{\Gamma}}=eA_{\Gamma}+m\frac{g_{\Gamma\Sigma}\omega^{\Sigma}}{\sqrt{g\left(\vec{\omega},\vec{\omega}\right)}}+...+\frac{S_{\Gamma\Sigma_1...\Sigma_n}\omega^{\Sigma_1}...\omega^{\Sigma_n}}{\left(S\left(\omega,...,\omega\right)\right)^{1-1/n}}+...,\\
\pi_{\alpha} &= p_{\alpha}-eA_{\alpha}-...\frac{S_{\alpha\beta_1...\beta_n}v^{\beta_1}...v^{\beta_n}}{\left(S\left(v,...,v\right)\right)^{n/(n+1)}}...=m\frac{g_{\alpha\beta}v^{\beta}}{\sqrt{g\left(\vec{v},\vec{v}\right)}}.
\end{aligned}
$$

In the second equation we have used v instead of ω for simplicity. Notice that this generalized momentum (π) is consistent with the usual quantum mechanical procedure $p\to p-eA$ that is used in Yang–Mills theories, as well as with the usual GR expression $p_{\alpha}=mg_{\alpha\beta}v^{\beta}$. Now it is easy to recognize the mass-shell constraint as a mathematical identity:

$$
\frac{\vec{v}}{\sqrt{\vec{v}^2}}\cdot\frac{\vec{v}}{\sqrt{\vec{v}^2}}=1\Rightarrow\pi_{\alpha}\pi^{\alpha}=m^2\Rightarrow\left(\vec{p}-e\vec{A}-\vec{S}_3\left(v\right)-\vec{S}_4\left(v\right)-...\right)^2\Psi=m^2\Psi.
$$

Notice that "gravity" as represented by the metric is gone, while the Klein–Gordon equation appears. The v dependence in the S terms reminds us about the problem related to the change of coordinates $(x,v)\to(x,p)$. So, at this stage we may proceed with the Klein–Gordon equation, if we wish.

Dirac equation from H=0. An interesting approach to the Dirac equation has been suggested by H. Rund.[4] The idea uses Hamiltonian linear in the momentum $(H=\gamma^{\alpha}p_{\alpha})$ and the base manifold principle group G. To have the Hamiltonian H invariant under G-transformations, the γ objects should transform appropriately and provide also a realization of the generators of G. Since we want γ and p to transform as vectors, it is clear that p should be a covariant derivative, but what is its structure? Consider a homogeneous Lagrangian that can be written as $L\left(\phi,\omega\right)=\omega^{\Gamma}p_{\Gamma}=\omega^{\Gamma}\partial L\left(\phi,\omega\right)/\partial\omega^{\Gamma}$ with a Hamiltonian function that is identically zero: $h=\omega^{\Gamma}\partial L\left(\phi,\omega\right)/\partial\omega^{\Gamma}-L\left(\phi,\omega\right)\equiv 0$. Notice that ω^{Γ} is the determinant of a matrix (the Jacobian of a transformation[21]); thus $\omega^{\Gamma}\to\gamma^{\Gamma}$ seems an interesting option for quantization. Even more, for the Dirac theory we know that γ^{α} are the 'velocities' $(dx/d\tau=\partial H/\partial p)$.

If we quantize using $(h \to H)$, then the space of physical states should satisfy: $H\Psi = 0$. By applying $\omega^\Gamma \to \gamma^\Gamma$, which means that the (generalized) velocity is considered as a vector with non-commutative components, we have $\left(\gamma^\Gamma p_\Gamma - L(\phi, \gamma)\right)\Psi = 0$. For a point particle, using the canonical form of the Lagrangian (4) and the algebra of the γ matrices following Run's approach[4] this gives:

$$\begin{aligned} H &= \gamma^\alpha p_\alpha - L(\phi, \gamma) = \gamma^\alpha p_\alpha - eA_\alpha \gamma^\alpha - m\sqrt{g_{\alpha\beta}\gamma^\alpha \gamma^\beta} - \ldots \sqrt[m]{S_m(\vec{\gamma}, \ldots, \vec{\gamma})}, \\ &\to \gamma^\alpha p_\alpha - eA_\alpha \gamma^\alpha - m - \ldots \sqrt[2m]{S_{2m} g^m} - \ldots \sqrt[2n+1]{S_{2n+1} g^n \gamma} \ldots \end{aligned}$$

Since $g_{\alpha\beta}$ is a symmetric tensor such that $\{\gamma^\alpha, \gamma^\beta\} \sim g^{\alpha\beta}$, then $g_{\alpha\beta}\gamma^\alpha\gamma^\beta \sim g_{\alpha\beta}\{\gamma^\alpha, \gamma^\beta\} \sim g_{\alpha\beta} g^{\alpha\beta} \sim 1$. Therefore, gravity seems to leave the picture again. The symmetric structure of the extra terms S_m can be used to reintroduce g and to reduce the powers of γ. Thus the higher even order terms contribute to the mass m, making it variable[22] with $\vec{x}$.

Summary. We have discussed the structure of the matter Lagrangian for extended objects. Imposing reparametrization invariance of the action S naturally leads to a first order homogeneous Lagrangian. In its canonical form, L contains electromagnetic and gravitational interactions, as well as interactions that are not yet identified. The non-relativistic limit for a brane has been defined as those coordinates where the brane is an integral sub-manifold of the target space. This gauge can be used to remove reparametrization invariance of the action S and make the Hamiltonian function suitable for canonical quantization. The existence of a mass-shell constraint is universal. It is essentially due to the gravitational (quadratic in velocities) type interaction in the Lagrangian and always leads to a Klein–Gordon like equation. Once the algebraic properties of the γ-matrices are defined, one can use $v \to \gamma$ quantization in the Hamiltonian function $h = pv - L(x, v)$ to obtain the Dirac equation.

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